Improved Carrier Tracking by Smoothing Estimators

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Smoothing as a way to improve the carrier phase estimation is proposed and analyzed. The performance of first- and second-order Kalman optimum smoothers are investigated. This performance is evaluated in terms of steady-state covariance error computation, dynamic tracking, and noise response. It is shown that with practical amounts of memory, a second-order smoother can have a position error due to an acceleration or jerk step input less than any prescribed maximum. As an example of importance to the NASA Deep Space Network, a second-order smoother can be used to track the Voyager spacecraft at Uranus and Neptune encounters with significantly better performance than a second-order phase-locked loop.

I. Introduction

Carrier tracking (carrier phase estimation) is traditionally accomplished using phase-locked loops (PLLs), which may be residual carrier tracking loops, Costas loops, or sideband-aided loops (combination of residual carrier tracking and Costas loops). All these loops are causal, i.e., only past and present data are used to estimate phase at the present time. It is reasonable to expect an improved estimation by using future data, i.e., by using noncausal filters. Implementing this kind of estimator requires storage of the data for the length of time necessary to acquire the future data. This is now feasible for DSN carrier tracking, using memory densities available with current technology.

Linear estimators are classified as performing prediction, filtering, or smoothing according to whether the parameter estimates for the present time are based on past data only, past and present data, or past, present, and future data. PLLs are causal; analog loops normally perform filtering, and digital or sampled data loops perform prediction, because of the transport lag inherent in the sampled data implementation.

The sources of tracking errors for a given estimator can be classified as those due to observation noise, modeled and unmodeled state noise, and unmodeled dynamics. In a DSN receiver, the observation noise is primarily due to receiving system noise; the state noise is due to the oscillator, transmitter, receiver, and propagation media instabilities; and the unmodeled dynamics are due to spacecraft acceleration, Earth rotation, or to the error in modeling these effects. By using a smoothing estimator, one might intuitively expect to reduce the phase-error variance due to observation noise by a factor of two, perhaps by using future data in a similar manner to past data. As shown later, this is true for first-order systems, but significantly more improvement is possible for higher-order systems. Finally, some cases of potential interest to the DSN advanced receiver are considered in this paper.

II. Practical Realization of the Smoothing Estimator

The proposed approach uses a Kalman estimation procedure as a method to extend the filter solution to a smoothing technique. Since the assumptions made to derive the Kalman solution may be different from the real case, it is necessary to evaluate the performance of the estimators for other cases. In this paper, in addition to the steady-state Kalman solution, a steady-state dynamic response and noise analysis are also presented.

Figure 1 shows a block diagram of a possible implementation of a smoothing estimator. The implementation shown is for a suppressed carrier signal with Costas-type phase detection. Residual carrier systems are similar except for the phase detection. The signal $r(t)$ is

$$r(t) = D(t) \cos (\omega_c t + \phi_c(t)) + v'(t) \tag{1}$$

where

$D(t)$ = binary data modulation

$\phi_c(t)$ = carrier phase

$v'(t)$ = narrowband white noise process

ω_c = received frequency

Also in Fig. 1,

$\theta(t)$ = phase error, $\theta(t) = \phi_c(t) - \phi_0(t)$

$\phi_0(t)$ = carrier phase estimate provided by the Costas Loop

$v''(t)$ = assumed white noise process

$v(n)$ = sample of a white noise process

$\hat{\phi}_c(n)$ = smoother estimate of the carrier phase

The suppressed carrier waveform $r(t)$ is initially tracked by the Costas loop, which provides the estimate $\phi_0(t)$. The input to the smoother is the sampled carrier phase $\phi_c(n)$ plus a noise term, $v(n)$, that is related to $v'(t)$. $\phi_c(n)$ is modeled as a state noise process driven by random noise, but actually also has variations due to unmodeled dynamics. The output of the smoother is an improved, albeit delayed, estimation of $\phi_c(n)$. Note that whereas the actual sampling of the signal is performed at the data symbol rate, T_s, the estimates $\theta(n)$ and $\phi_0(n)$ are based on the averages of those samples; i.e., the sampling interval for $\phi_c(n)$ is $T = M\, T_s$.

In the proposed implementation, the initial phase estimates are made at the channel symbol rate, but are then averaged over M symbols before application to the real-time PLL and to the smoother. The smoother update rate of $1/M$ times the symbol rate can be chosen appropriately for the system parameters. A rate of 10 to 20 times the bandwidth of the filter is typical, i.e., of the filter that forms the basis of the smoother. As shown later, the smoother delay should be several correlation times. Suppose the delay is 0.5 s, which might be typical for DSN carrier tracking. At a symbol rate of 60 k symbols/s, a high rate for Voyager, only 30,000 symbols need be stored, which is practical with current memories.

III. Smoother Mathematical Model and General Solution

This section presents the mathematical model and the general solutions to the filtering and smoothing problems. Then the steady-state solutions of the smoother are investigated for the first- and second-order cases in Sections IV and V.

In navigation problems, the phase is related to position and the phase rate to velocity. Also, to use the terminology in several of the references, the following change of variables is used in the remainder of this article. For the second-order system, define the state vector $\mathbf{x}(n)$ at sampling time nT as

$$\mathbf{x}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} \tag{2}$$

where

$$x_1(n) = \phi_c(n) = \text{position}$$

$$x_2(n) = \dot{\phi}_c(n) = \text{velocity}$$

We also consider a first-order system that has only the position variable.

A. Nomenclature

The following notation and assumptions are to be used:

$\mathbf{\Phi}$ = state transition matrix (assumed constant)

$\mathbf{H}$ = observation matrix (assumed constant)

$\mathbf{K}(n)$ = Kalman gain matrix

$\mathbf{P}_F(n)$ = filter error covariance

$\mathbf{P}_P(n)$ = predictor error covariance

$\mathbf{P}_S(n)$ = smoother error covariance

$\mathbf{v}(n)$ = observation noise sample vector (white)

$\mathbf{w}(n)$ = state noise sample vector (white)

$\mathbf{x}(n)$ = state vector

$\mathbf{y}(n)$ = measurement vector

$\mathbf{Q}$ = covariance of the process $\mathbf{w}(n)$ (assumed constant)

$\mathbf{R}$ = covariance of the process $\mathbf{v}(n)$ (assumed constant)

The smoother proposed is an optimum linear Kalman smoother. Since the various equations that describe the estimator are given in terms of the Kalman filter equations, some of these equations are repeated here; however, the detailed description and solutions of them are left to the references. Noting the assumptions stated above, the following system model equations are considered:

$$\mathbf{x}(n+1) = \Phi\, \mathbf{x}(n) + \mathbf{w}(n)$$

$$\mathbf{Q} = E\,[\mathbf{w}(n)\mathbf{w}^T(n)] \tag{3}$$

$$y(n) = \mathbf{H}\, \mathbf{x}(n) + \mathbf{v}(n)$$

$$\mathbf{R} = E\,[\mathbf{v}(n)\mathbf{v}^T(n)] \tag{4}$$

B. Filter Equations

For the filter estimation, i.e., calculation of the estimate $\mathbf{x}_F(n)$ of $\mathbf{x}(n)$ using observations up to the present sampling time, n, the general Kalman filter equations are (Ref. 1)

filter state:

$$\mathbf{x}_F(n) = \Phi\, \mathbf{x}_F(n-1) + \mathbf{K}(n)\, [\mathbf{y}(n) - \mathbf{H}\Phi\, \mathbf{x}_F(n-1)] \tag{5}$$

prediction error covariance matrix:

$$\mathbf{P}_P(n) = \Phi\, \mathbf{P}_F(n-1)\, \Phi^T + \mathbf{Q} \tag{6}$$

filter error covariance matrix:

$$\mathbf{P}_F(n) = [\mathbf{I} - \mathbf{K}(n)\mathbf{H}]\, \mathbf{P}_P(n) \tag{7}$$

Kalman gain matrix:

$$\mathbf{K}(n) = \mathbf{P}_P(n)\, \mathbf{H}^T\, [\mathbf{H}\mathbf{P}_P(n)\, \mathbf{H}^T + \mathbf{R}]^{-1} \tag{8}$$

The initial conditions are given as

$$E\,[\mathbf{x}(0)] = \mathbf{x}_F(0)$$

$$E\,[(\mathbf{x}(0) - \mathbf{x}_F(0))\,(\mathbf{x}(0) - \mathbf{x}_F(0))^T] = \mathbf{P}_F(0)$$

C. Smoother Equations

The following equations are for the fixed-lag smoother; i.e., a smoothing solution that estimates $\mathbf{x}(n)$ on the basis of measurements up to time $n + N$. From Refs. 1 and 2, the equations for such an estimate, denoted $\mathbf{x}_S(n)$, are

$$\begin{aligned}\mathbf{x}_S(n+1) = {} & \Phi\, \mathbf{x}_S(n) + \mathbf{Q}\Phi^T\, \mathbf{P}_F^{-1}\, [\mathbf{x}_S(n) - \mathbf{x}_F(n)] \\ & + \mathbf{B}(n+1+N)\, \mathbf{K}(n+1+N)\, [\mathbf{y}(n+1+N) \\ & - \mathbf{H}\Phi\, \mathbf{x}_F(n+N)]\end{aligned} \tag{9}$$

where

$$\mathbf{B}(n+1+N) = \prod_{i=n+1}^{n+N} \mathbf{A}(i), \quad \mathbf{A}(i) = \mathbf{P}_F(i)\Phi^T\, \mathbf{P}_P^{-1}\,(i+1)$$

$$n = 0, 1, \cdots$$

The initial state is $\mathbf{x}_S(0)$. The error covariance matrix of the smoothed estimates is

$$\begin{aligned}\mathbf{P}_S(n+1) = {} & \mathbf{P}_P(n+1) - \mathbf{B}(n+1+N)\, \mathbf{K}(n+1+N)\, \mathbf{H}\, \mathbf{P}_P(n \\ & + 1 + N)\, \mathbf{B}^T\,(n+1+N) - \mathbf{A}^{-1}(n)\, [\mathbf{P}_F(n) \\ & - \mathbf{P}_S(n)]\, (\mathbf{A}^T(n))^{-1}\end{aligned} \tag{10}$$

for $n = 0, 1, 2, \cdots$, where the initial condition is $\mathbf{P}_S(0)$. Both initial conditions, $\mathbf{x}_S(0)$ and $\mathbf{P}_S(0)$, are computed by using the fixed-point smoother equations described next (Refs.1 and 2). Let $\mathbf{x}_S(0\,|\,i)$ denote the optimum smoothing solution for the initial time 0 when i measurements have been taken. Then

$$\mathbf{x}_S(0\,|\,i) = \mathbf{x}_S(0\,|\,i-1) + \mathbf{B}(i-1)\, [\mathbf{x}_F(i) - \Phi\, \mathbf{x}_F(i-1)] \tag{11}$$

where

$$\mathbf{B}(i-1) = \prod_{j=0}^{i-1} \mathbf{A}(i), \qquad \mathbf{A}(i) = \mathbf{P}_F(j)\Phi^T\, \mathbf{P}_P^{-1}\,(j+1)$$

$$\mathbf{x}_S(0\,|\,0) = \mathbf{x}_F(0)$$

$$\mathbf{P}_S(0|i) = \mathbf{P}_S(0|i-1) + \mathbf{B}(i-1)\,[\mathbf{P}_F(i) - \mathbf{P}_P(i)]\,\mathbf{B}^T(i-1)$$

$$\mathbf{P}_S(0|0) = \mathbf{P}_F(0) \tag{12}$$

The initial conditions for the smoother with delay N are then $\mathbf{x}_S(0) = \mathbf{x}_S(0|N)$ and $\mathbf{P}_S(0) = \mathbf{P}_S(0|N)$.

The steady-state solutions of the first- and second-order smoothers are investigated in Sections IV and V.

IV. First-Order Smoother

In this case we consider only the estimation and measurement of position. Thus all the variables are scalars:

$$\begin{aligned} x &= \text{position} \\ \Phi &= 1 \\ \mathbf{H} &= 1 \\ \mathbf{Q} &= qT = \sigma_v^2\,T^2 \\ \mathbf{R} &= \sigma_0^2 \end{aligned} \tag{13}$$

where T is the sampling interval, q is the spectral density of the continuous white velocity process, and σ_0^2 is the variance of the position measurement.

Define the parameter $s = \sigma_0/(\sigma_v T)$. The value of s completely characterizes the solution. For the steady-state solution, $\mathbf{P}_F(n) = \mathbf{P}_F(n-1) = \mathbf{P}_F$, $\mathbf{P}_P(n) = \mathbf{P}_P(n-1) = \mathbf{P}_P$ and $\mathbf{K}(n) = \mathbf{K}(n-1) = \mathbf{K}$. Solving the corresponding Eqs. (6 through 8), the following solutions are obtained for the filter estimator, where P_P, P_F, P_S, and K are the scalar values of the corresponding matrices:

$$\frac{P_P}{\sigma_0^2} = \frac{\sqrt{1/4 + s^2} + 1/2}{s^2} \tag{14}$$

$$\frac{P_F}{\sigma_0^2} = \frac{\sqrt{1/4 + s^2} - 1/2}{s^2} \tag{15}$$

$$K = \frac{1}{\sqrt{1/4 + s^2} + 1/2} \tag{16}$$

$$\frac{P_S}{P_F} = \frac{1}{1+A} + \frac{KA^{2N+1}}{1-A^2} \tag{17}$$

where

$$A = P_F P_P^{-1} = 1 - K.$$

It is interesting to observe that as N increases, P_S/P_F approaches $(1 + A)^{-1}$. If also s is large, P_S/P_F approaches 1/2. Thus the variance of position error is reduced by a factor of 2 for the assumed conditions. This is in accordance with intuition as stated in the introduction. These results say nothing about the performance for other conditions.

A. Dynamic Tracking Performance

The dynamic tracking error of an estimation system is an important performance measure that is not obtained from the Kalman analysis. To obtain this performance, a z-transform approach is used.

Assuming steady-state values for the various parameters and taking the z-transform of the corresponding equations, the following transfer functions are obtained. For filtering,

$$F_F(z) = \frac{X_F(z)}{y(z)} = \frac{Kz}{z - (1-K)} \tag{18}$$

$$E_F(z) = \frac{y(z) - X_F(z)}{y(z)} = \frac{(z-1)(1-K)}{z-(1-K)} \tag{19}$$

where $E_F(z)$ is the error transfer function of the filter.

The smoother and smoother error transfer functions are

$$F_S(z) = \frac{X_S(z)}{y(z)} = \frac{z\,[A^N K z^{N+1} - A^N K z^N - 1/s^2]}{(z - 1/A)(z - A)} \tag{20}$$

$$E_S(z) = \frac{(z-1)(z - 1 - A^N K z^{N+1})}{(z-1/A)(z-A)} \tag{21}$$

1. Response to step velocity. The steady-state position error due to a 1-m/s velocity step input can be computed by using the final value theorem. For this kind of input, $Y(z) = Tz\,(z-1)^{-2}$ and the position errors for filtering and smoothing are

$$e_F(\infty) = \frac{(1-K)\,T}{K} \tag{22}$$

$$e_S(\infty) = \frac{(1-K)^{N+1}\,T}{K} \tag{23}$$

The importance of this result is that the error of a first-order smoother to a step velocity input can be made arbitrarily small by choosing N sufficiently large since $e_S(\infty) \to 0$ as $N \to \infty$.

2. **Response to step acceleration.** For infinite delay smoothing, the error transfer function is ($N = \infty$ in Eq. (21)):

$$E_S(z) = \frac{(z-1)^2}{\left(z - \frac{1}{A}\right)(z - A)} \tag{24}$$

The z-transform of a 1-m/s^2 acceleration step input is $Y(z) = T^2 z(z+1)(z-1)^{-3}/2$. The position error for infinite delay smoothing is

$$e_S(\infty) = \lim_{z \to 1} \frac{(z-1)}{z} E_S(z)\, Y(z) = -\frac{(1-K)\, T^2}{K^2} \tag{25}$$

For $s > 5$ this error reduces approximately to

$$e_S(\infty) \approx -s^2 T^2 \tag{26}$$

For finite N, the error due to an acceleration step increases linearly with time. Figure 2 illustrates the position error as a function of time when N is finite and the input is a 1-m/s^2 acceleration step. The value of s is 5. It can be deduced from the results in Fig. 2 that for N finite, the error due to an acceleration step increases with time (t), approximately as $e_S(\infty)t$ where $e_S(\infty)$ is the position final error due to a 1-m/s velocity step and is given by Eq. (23). Thus for a known maximum observation time t, the error due to step acceleration can be made arbitrarily small by choosing N sufficiently large.

B. Noise Response

The Kalman solution gives the estimate error covariance due to the state noise and the observation noise. To obtain the error covariance due to the observation noise only, a z-transform approach is again used. The noise response of the kind of linear estimator discussed here is given by

$$\left(\frac{\sigma^2_{out}}{\sigma^2_{in}}\right) = \frac{1}{2\pi j} \oint_{\substack{UNIT\\CIRCLE}} dz\, F(z)\, F(z^{-1})\, z^{-1} \tag{27}$$

Substituting for $F_F(z)$ from Eq. (18) into Eq. (27), the noise reduction of the filter is

$$\left(\frac{\sigma^2_{out}}{\sigma^2_{in}}\right)_F = \frac{K}{2-K} \tag{28}$$

Computing Eq. (27) for the smoother usually requires some kind of numerical method unless $N \to \infty$, in which case

$$\left(\frac{\sigma^2_{out}}{\sigma^2_{in}}\right)_S = \frac{(1-K)^2\,(1+(1-K)^2)}{s^4\,(1-(1-K)^2)^3} \tag{29}$$

For this case of no state noise, Fig. 3 shows $(\sigma^2_{out}/\sigma^2_{in})$ vs NT/τ_1, where τ_1 is the time constant (correlation time) of the filter. Of course the filter and smoother parameters are based on the nonzero state covariance. The time constant is

$$\tau_1 = -\frac{T}{\ln(1-K)} \tag{30}$$

Note that K depends on s. It can be observed in Fig. 3 that if the smoothing time lag NT is approximately $4\tau_1$, then the noise response approximately reaches its asymptotic value given by Eq. (29).

V. Second-Order Smoother

For the second-order case, the state variable is a vector as in Eq. (2) and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{H} = [1 \quad 0] \tag{31}$$

$$\mathbf{Q} = qT \begin{bmatrix} T^2/3 & T/2 \\ T/2 & 1 \end{bmatrix}, \qquad q = \sigma_0^2\, T$$

$$\mathbf{R} = R = \sigma_0^2$$

where q is the spectral density of the assumed continuous acceleration process, σ_0^2 is the variance of the position measurement, and σ_a^2 is the variance of the acceleration noise. The covariance matrix $\mathbf{Q}$ is chosen to be consistent with previous work (Ref. 3).

The steady-state solution for the filter gains and error covariance matrices are obtained after considerable algebraic manipulations and can be found in Ref. 3. In Ref. 3, all the solutions are expressed in terms of two parameters denoted there as r and s. For the case presented in this paper, s is infinity since there is no measurement of velocity. Therefore, all the solutions are functions of r defined as

$$r = \frac{4\sigma_0}{\sigma_a T^2} \tag{32}$$

From Eq. (10), the smoother steady-state error covariance $\mathbf{P}_S$ is the solution of

$$\mathbf{P}_S = \mathbf{P}_P - \mathbf{B}\,\mathbf{K}\,\mathbf{H}\,\mathbf{P}_P\,\mathbf{B}^T - \mathbf{A}^{-1}\,[\mathbf{P}_F - \mathbf{P}_S]\,(\mathbf{A}^T)^{-1} \tag{33}$$

where $\mathbf{P}_P$, $\mathbf{P}_F$, $\mathbf{K}$ are the steady-state filter solutions (Refs. 3 and 4). The asymptotic improvement due to smoothing $(\mathbf{P}_F - \mathbf{P}_S)$ may be shown to be (Ref. 5)

$$\mathbf{P}_F - \mathbf{P}_S = \mathbf{P}_F \sum_{i=0}^{N} \{[\widetilde{\Phi}^T]^i\,\mathbf{H}^T\,[\mathbf{H}\,\mathbf{P}_F\,\mathbf{H}^T + \mathbf{R}]^{-1}\,\mathbf{H}\widetilde{\Phi}^i\}\,\mathbf{P}_F \tag{34}$$

where $\widetilde{\Phi} = \Phi - \mathbf{KH}$. The smoothing gain is obviously a monotonic function of N with most of the gain realized within a few time constants of the filter. Letting N approach infinity, we obtain

$$[\mathbf{P}_F - \mathbf{P}_{S,\infty}] - \mathbf{P}_F\,\widetilde{\Phi}^T\,\mathbf{P}_F^{-1}\,[\mathbf{P}_F - \mathbf{P}_{S,\infty}]\,\mathbf{P}_F^{-1}\,\widetilde{\Phi}\,\mathbf{P}_F = \mathbf{P}_F\,\mathbf{H}^T\,\left[\mathbf{H}\mathbf{P}_F\,\mathbf{H}^T + \mathbf{R}\right]^{-1}\,\mathbf{H}\mathbf{P}_F \tag{35}$$

The above is a linear equation in the elements of $(\mathbf{P}_F - \mathbf{P}_{S,\infty})$ and can be solved explicitly. Also, $\mathbf{A} = \mathbf{P}_F\,\Phi^T\,\mathbf{P}_P^{-1}$ and $\mathbf{B} = \mathbf{A}^N$.

$\mathbf{P}_S$ as a function of r and the delay N is determined by Eq. (33). Figure 4 shows the smoothed position accuracy, $P_S(1, 1)/\sigma_0^2$, as a function of r and N. The case $N = 0$ corresponds to the filter solution. For large observation noise compared to process noise (large r), the mean square position accuracy improvement due to smoothing over filtering is almost a factor of 4 for large N ($N \geqslant 20$). This is almost twice the improvement realized in the first-order case. Intuitively, the additional improvement for the second-order case comes from the implicit estimation of rate as well as position.

As in the first-order case, dynamic tracking performance and noise response analyses are presented next.

A. Dynamic Tracking Performance

Taking the z-transforms of the corresponding recursive equations, the following transfer functions are obtained for the filter and smoother estimators.

$$\frac{X_{F1}(z)}{Y(z)} = F_F(z) = \frac{K_1 z^2 + (TK_2 - K_1)\,z}{z^2 + (K_1 + K_2 T - 2)\,z + (1 - K_1)} \tag{36}$$

$$\frac{X_{S1}(z)}{Y(z)} = F_S(z) = F_F(z) + \frac{z(z-1)^2\,\mathbf{H}\,[\mathbf{I} - z^N\mathbf{A}^N]\,[\mathbf{I} - \mathbf{A}z]^{-1}\mathbf{AK}}{z^2 + (K_1 + K_2 T - 2)z + (1 - K_1)} \tag{37}$$

where $\mathbf{K} = (K_1, K_2)^T$ is the Kalman gain vector, and $X_{F1}(z)$ and $X_{S1}(z)$ stand for the first elements of the vectors $\mathbf{X}_F(z)$ and $\mathbf{X}_S(z)$, respectively. Equation (37) has been derived from a numerically more stable recursive equation than of Eq. (9). This relation is

$$\mathbf{x}_S(n) = \mathbf{x}_F(n) + \mathbf{A}[\mathbf{x}_F(n+1) - \Phi\mathbf{x}_F(n)] + \cdots + \mathbf{A}^N[\mathbf{x}_F(n+N) - \Phi\mathbf{x}_F(n+N-1)] \tag{38}$$

The error transfer functions are

$$E_F(z) = \frac{(z-1)^2\,(1 - K_1)}{(z-1)^2 + (K_1 + K_2 T)(z-1) + K_2 T} \tag{39}$$

$$E_S(z) = E_F(z) - F_S(z) = \frac{(z-1)^2\,[1 - K_1 - z\mathbf{H}\,[\mathbf{I} - z^N\,\mathbf{A}^N]\,[\mathbf{I} - \mathbf{A}z]^{-1}\mathbf{AK}]}{(z-1)^2 + (K_1 + K_2 T)(z-1) + K_2 T} \tag{40}$$

1. **Response to step acceleration.** The z-transform of a 1-m/s^2 acceleration step input is $Y(z) = T^2 z(z+1)(z-1)^{-3}/2$. The steady-state position errors for the filter and smoother are

$$e_F(\infty) = \frac{(1 - K_1)T}{K_2} \tag{41}$$

$$e_S(\infty) = \frac{(1 - K_1 - \mathbf{H}[\mathbf{I} - \mathbf{A}^N]\,[\mathbf{I} - \mathbf{A}]^{-1}\mathbf{AK})T}{K_2} \tag{42}$$

The filter has finite but nonzero error due to step acceleration, but the smoother error approaches zero for large N.

Figure 5 shows the steady-state position error computed from Eqs. (41) and (42) vs the normalized smoothing time lag, NT/τ_2. The time constant, τ_2, of the second-order Kalman filter, Eq. (35), is obtained by using the mapping between the s plane and z plane (Ref. 6). The time constant is

$$\tau_2 = -\frac{2T}{\ln(1 - K_1)} \tag{43}$$

In Fig. 5, note the oscillatory nature of $|e_S(\infty)|$. One should not design N to operate on a null of this function because differences between the model and the actual system will affect the locations of the nulls. Arbitrarily small error can be achieved by choosing N large enough so that the error is sufficiently small for this N and all larger N.

2. **Response to step jerk.** For infinite delay smoothing, the position error transfer function is ($N \to \infty$ in Eq. (40))

$$E_S(z) = \frac{(z-1)^4 (1-K_1)}{[(z-1)^2 + (K_1 + K_2T)(z-1) + K_2T]\,[(z-1)^2 - K_1 z(z-1) + K_2 z]} \tag{44}$$

The z-transform of a 1-m/s^3 step jerk is

$$Y(z) = \frac{z(z^2 + 4z + 1)}{(z-1)^4} T^3 \tag{45}$$

From Eqs. (44) and (45) and the final value theorem, it is easy to see that in this case $e_S(\infty) \to 0$. Thus, for infinite smoothing, the steady-state error to step jerk is zero.

For finite N, the error due to a step jerk increases linearly with time. Figure 6 shows the position error as a function of time when N is finite. The value of r is 500. It can be seen in Fig. 6 that for N finite, the error due to a step jerk increases with time approximately as $e_S(\infty)t$ where $e_S(\infty)$ is the position final error due to a 1-m/s^2 acceleration step and is given by Eq. (42). This error can be made arbitrarily small for any finite time interval.

3. **Summary of dynamic responses.** Table 1 shows a summary of the dynamic responses for the first- and second-order Kalman filters and smoothers.

B. Noise Response

As in the first-order case, Eq. (27), the noise response of the second order Kalman filter with no state noise is given by

$$\left(\frac{\sigma^2_{out}}{\sigma^2_{in}}\right)_F = \frac{2K_2T - 3K_1K_2T + 2K_1^2}{K_1(4 - 2K_1 + K_2T)} \tag{46}$$

The smoother responses are given in Eqs. (34) and (35). From Eq. (37), a specific expression for the smoother transfer function for $N = \infty$ is

$$F_S(z) = \frac{z[(2K_2 - K_2K_1 - K_1^2)(z-1)^2 + K_2^2 z]}{[(z-1)^2 + (K_1 + K_2)(z-1) + K_2]\,[(z-1)^2 - (K_1 + K_2)z(z-1) + K_2 z^2]} \tag{47}$$

Replacing $F_S(z)$ in Eq. (27) and using the results given in Ref. 7, the noise response can be easily computed.

Figure 7 shows the noise response of the Kalman filter and smoother vs NT/τ_2. Note that for any value of the parameter r, a smoothing time delay of roughly 1.5 τ_2 is enough for the smoother to approach its asymptotic noise response value.

VI. Examples for Uranus and Neptune Encounter

Of particular interest in the DSN is the tracking of the Voyager spacecraft in its close encounters with Uranus and Neptune. The approximate dynamics for Voyager encounters are

Planet	Acceleration	Jerk
Uranus	-0.32 m/s^2	0.83×10^{-4} m/s
Neptune	-4.00 m/s^2	0.29×10^{-2} m/s^3

Assuming an X-band carrier frequency of 8.4 GHz, a desired phase error of less than 1° corresponds approximately to $e_S(\infty) = 10^{-4}$m. In this example, it is assumed that the one-sided equivalent noise bandwidth of the Kalman filter used by the smoother is $B_F = 5$ Hz, because the Voyager spacecraft oscillator is known to be stable enough for this loop bandwidth. Also, it is assumed that $B_FT = 0.05$, so $T = 0.01$ s. The assumed Kalman parameters r and s are determined from the noise results of the previous sections and from the relationship $\sigma^2_{out}/\sigma^2_{in} = 2B_FT$ for the filter ($N = 0$).

A. First-Order Smoother

For an acceleration step input and N finite, the first-order Kalman smoother has a position error increasing linearly with time (see Fig. 2). For $B_FT = 0.05$, the corresponding design parameter obtained from Fig. 3 is $s = 5$. In this case, from Eq. (26) or Fig. 2, even when N is very large the position error to a 1-m/s^2 acceleration step is approximately 0.0025 m, which means that the maximum tolerable position error for both Uranus and Neptune encounter is exceeded.

B. Second-Order Smoother

For $B_FT = 0.05$, the design parameter r obtained from Fig. 7 is $r = 500$. For Uranus encounter, a 10^{-4}-m lag error with a 0.32-m/s^2 acceleration corresponds to $e_S(\infty)/T = 0.031$ at 1 m/s^2, which requires $NT/\tau_2 \geqslant 3.4$ (Fig. 5). For Neptune encounter, $NT/\tau_2 \geqslant 6.3$.

C. Second-Order Filter

The smoother reduces to the filter for $N = 0$. This corresponds to use of a PLL. The steady-state delay error is 0.01 m for a 1-m/s^2 acceleration, or 0.003 m at Uranus and 0.04 m at Neptune. These errors are not satisfactory.

D. Conclusion for Voyager

For the example parameters, a second-order PLL with a one-sided loop bandwidth of 5 Hz will not yield adequate dynamic performance for the Voyager encounters. Neither will a first-order smoother. A second-order smoother will result in satisfactory performance.

VII. Comments on Stability and Implementation Constraints

The smoother-state estimation equation as in Eq. (37) explicitly shows the dependence of the smoother on the corresponding Kalman filter. Since this dependence is expressed in a form of a finite sum of filter estimates, it follows that if the filter is stable so is the smoother. There are a large number of methods to ensure the filter stability in actual practical implementations (Ref. 1). For the results presented in this paper, a word length of 64 bits has been used to implement the various filters. It was found that a 32-bit word length was not enough to guarantee the stability of the filters, but 64 bits do suffice.

VIII. Conclusions

Use of smoothing filters to improve the phase estimation in carrier tracking systems has been proposed. First-order and second-order Kalman optimum smoothers are analyzed. The steady-state filter gains and error covariances are computed. Dynamic tracking performance and noise response are investigated by means of z-transform techniques. It is shown that a second-order Kalman smoother can keep the position error due to step acceleration or step-jerk inputs less than any prescribed maximum, assuming the step jerk is applied for finite time. It is also shown that the phase-error variance due to observation noise (receiver noise) is reduced by almost a factor of four by use of second-order smoothing rather than second-order filtering or a second-order PLL.

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Table 1. Summary of dynamic responses—steady-state position error, m

	Step velocity V	Step acceleration A	Step jerk J
First-order filter	$\frac{V(1-K)T}{K}$	∞	∞
First-order smoother	$\frac{V(1-K)^{N+1}T}{K}$	∞ (Ramp, Fig. 2)	∞
First-order smoother, $N = \infty$	0	$\frac{-A(1-K)T^2}{K^2}$	∞
Second-order filter	0	$\frac{A(1-K_1)T}{K_2}$	∞
Second-order smoother	0	$A \cdot$ (Eq. (41))	∞ (Ramp. Fig. 6)
Second-order smoother, $N = \infty$	0	0	0

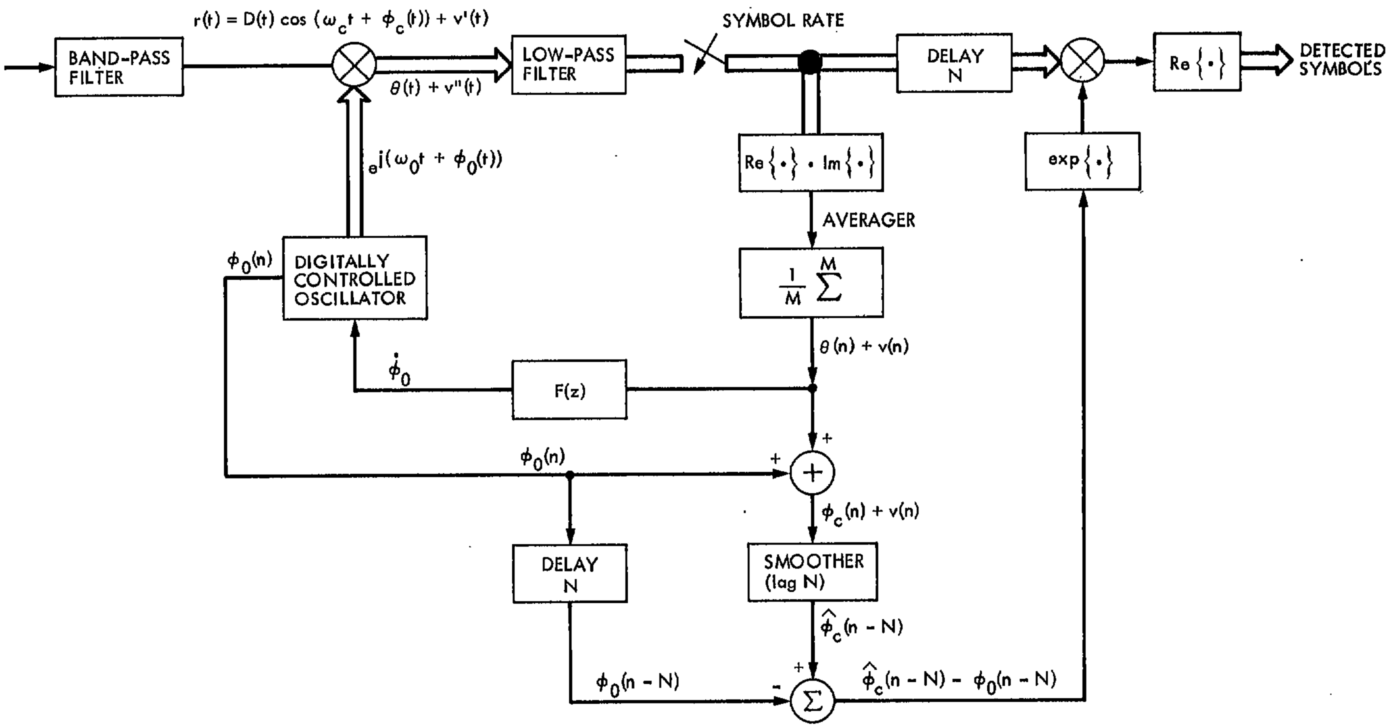

Fig. 1. Possible realization of smoothing estimator and data detection

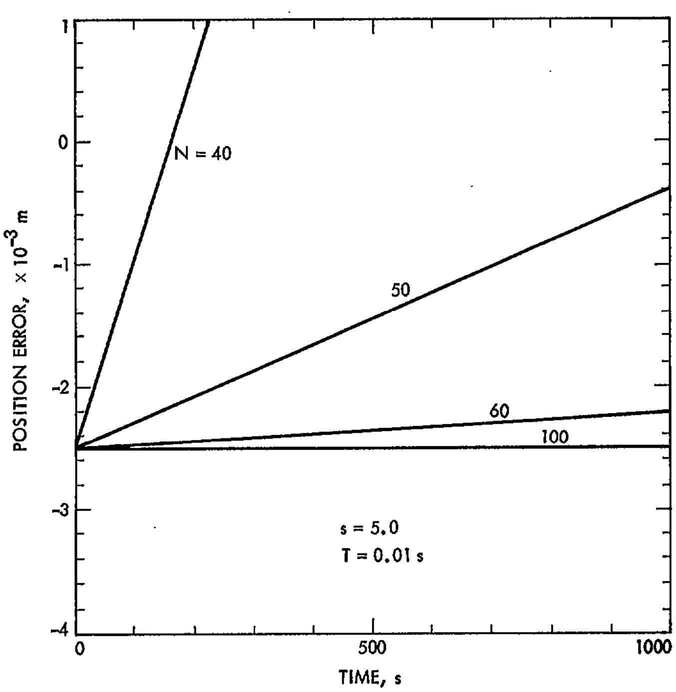

Fig. 2. First-order smoother: steady-state dynamic response to 1-m/s² step acceleration

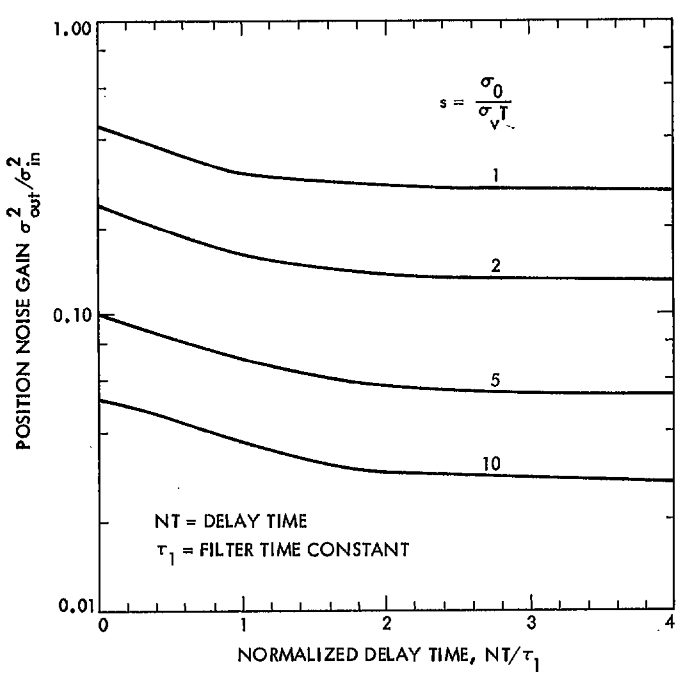

Fig. 3. First-order smoother response to observation noise; no state noise.

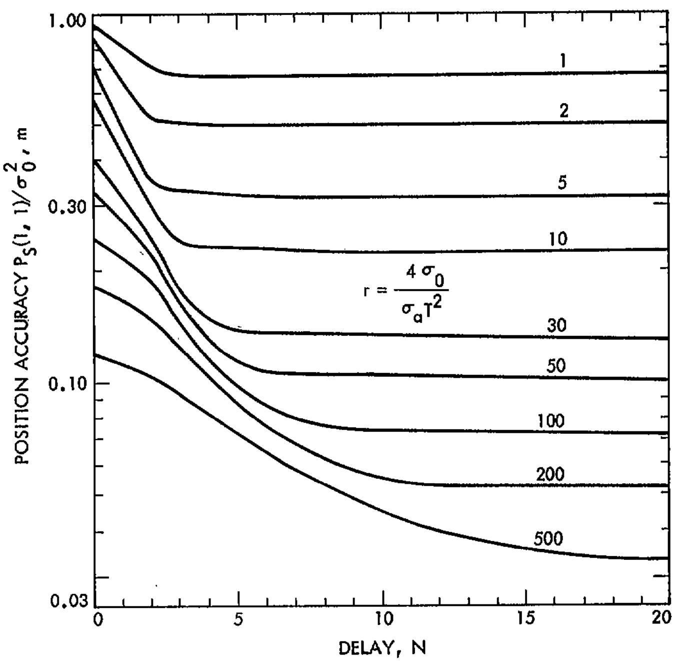

Fig. 4. Second-order smoother position accuracy with both observation and state noise

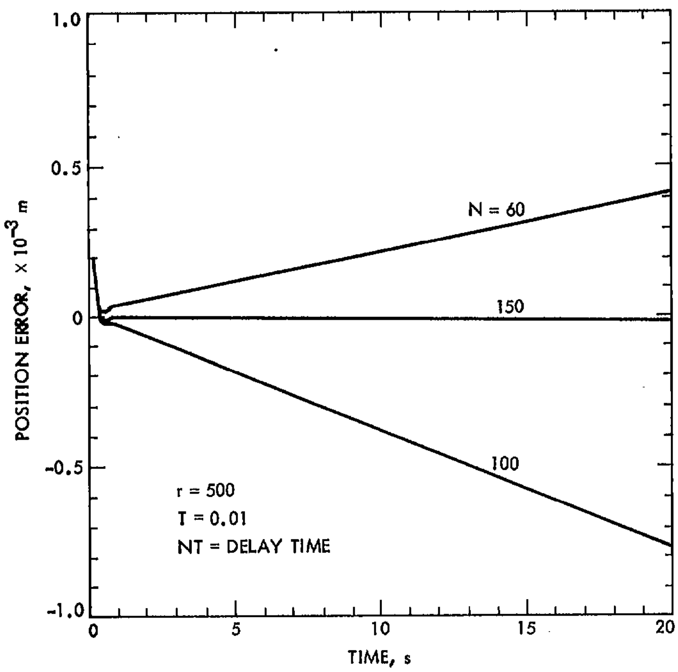

Fig. 6. Second-order smoother transient response to 1–m/s² step jerk

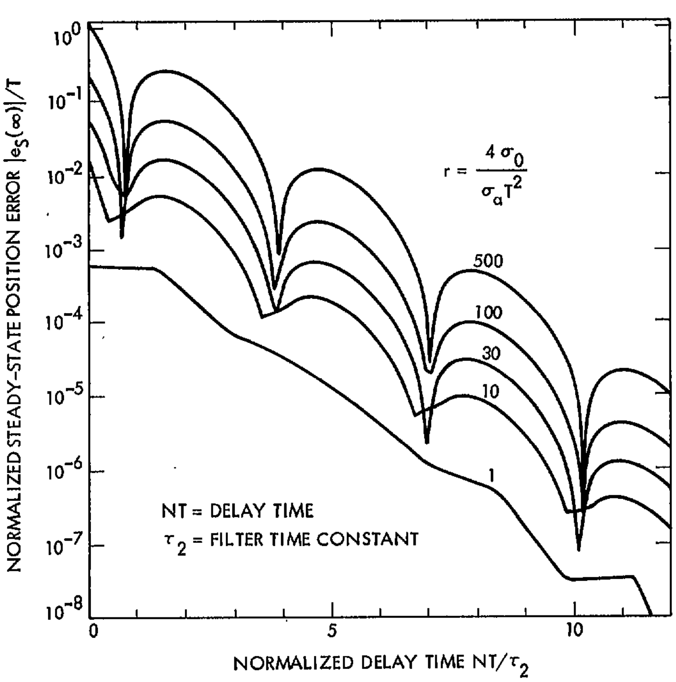

Fig. 5. Second-order smoother steady-state error to 1 m/s² step acceleration

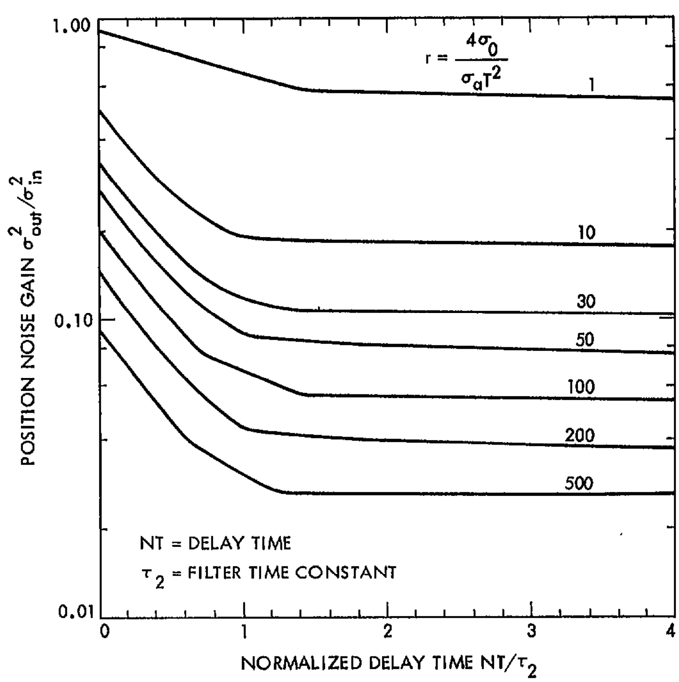

Fig. 7. Second-order smoother response to observation noise; no state noise